Proceedings of the ASME 2015 International Design and Engineering Technical Conferences & Computers and Information in Engineering Conference
IDETC/CIE 2015
August 2-5, 2015, Boston, Massachusetts, USA

DETC2015-47359

DESIGN SENSITIVITY METHOD FOR SAMPLING-BASED RBDO WITH FIXED COV

Hyunkyoo Cho and K. K. Choi*
Department of Mechanical & Industrial Engineering
College of Engineering, The University of Iowa
Iowa City, IA 52242, USA
hyunkyoo-cho@uiowa.edu, kyung-choi@uiowa.edu
*Corresponding author

Ikjin Lee
Korea Advanced Institute of Science and Technology (KAIST)
Daejeon, Republic of Korea 305-701
Ikjin.lee@kaist.ac.kr

David Lamb
US Army RDECOM/TARDEC
Warren, MI 48397-5000, USA
david.lamb@us.army.mil

ABSTRACT

Conventional reliability-based design optimization (RBDO) uses the means of input random variables as its design variables; and the standard deviations (STDEVs) of the random variables are fixed constants. However, the fixed STDEVs may not correctly represent certain RBDO problems well, especially when a specified tolerance of the input random variable is presented as a percentage of the mean value. For this kind of design problem, the coefficients of variations (COVs) of the input random variables should be fixed, which means STDEVs are not fixed. In this paper, a method to calculate the design sensitivity of probability of failure for RBDO with fixed COV is developed. For sampling-based RBDO, which uses Monte Carlo simulation for reliability analysis, the design sensitivity of the probability of failure is derived using a first-order score function. The score function contains the effect of the change in the STDEV in addition to the change in the mean. As copulas are used for the design sensitivity, correlated input random variables also can be used for RBDO with fixed COV. Moreover, the design sensitivity can be calculated efficiently during the evaluation of the probability of failure. Using a mathematical example, the accuracy and efficiency of the developed method are verified. The RBDO result for mathematical and physical problems indicates that the developed method provides accurate design sensitivity in the optimization process.

KEYWORDS

RBDO, Fixed COV, Sampling-based RBDO, Score Function, Tolerance of Random Variable

NOMENCLATURE

RBDO	reliability-based design optimization
STDEV	standard deviation
COV	coefficient of variation
PDF	probability density function
CDF	cumulative distribution function
MCS	Monte Carlo simulation
FDM	finite difference method
N	number of random variables
NC	number of constraints
NDV	number of design variables
X_i, $\mathbf{X}$	random variable, random variable vector, $\mathbf{X} = [X_1, \ldots, X_N]^T$
x_i, $\mathbf{x}$	realization of X_i and $\mathbf{X}$, $\mathbf{x} = [x_1, \ldots, x_N]^T$
d_i, $\mathbf{d}$	design variable, design variable vector, $\mathbf{d} = [d_1, \ldots, d_{NDV}]^T$
$G_j(\mathbf{X})$	j-th performance measure, $G_j(\mathbf{X}) > 0$ indicates failure.
$f_{\mathbf{X}}(\mathbf{x})$	joint PDF of $\mathbf{X}$
$f_{X_i}(x_i; a_i, b_i)$	marginal PDF of random variable X_i
$F_{X_i}(x_i; a_i, b_i)$	marginal CDF of random variable X_i
μ_i, σ_i, COV_i	mean, STDEV and COV of random variable X_i
a_i, b_i	two parameters for marginal PDF and CDF of random variable X_i
$S_{d_i}^{(1)}(\mathbf{x})$	(first-order) score function of design variable, $S_{d_i}^{(1)}(\mathbf{x}) \equiv \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x})}{\partial d_i}$

DISTRIBUTION STATEMENT A. Approved for public release; distribution is unlimited.

Report Documentation Page

Form Approved
OMB No. 0704-0188

Public reporting burden for the collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to a penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.

1. REPORT DATE **29 APR 2015**	2. REPORT TYPE	3. DATES COVERED **00-00-2015 to 00-00-2015**
4. TITLE AND SUBTITLE **DESIGN SENSITIVITY METHOD FOR SAMPLING-BASED RBDO WITH FIXED COV**		5a. CONTRACT NUMBER 5b. GRANT NUMBER 5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)		5d. PROJECT NUMBER 5e. TASK NUMBER 5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **US Army RDECOM-TARDEC,6501 E. 11 Mile Road,Warren,MI,48397-5000**		8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)		10. SPONSOR/MONITOR’S ACRONYM(S) 11. SPONSOR/MONITOR’S REPORT NUMBER(S)
12. DISTRIBUTION/AVAILABILITY STATEMENT **Approved for public release; distribution unlimited**		
13. SUPPLEMENTARY NOTES		
14. ABSTRACT **See Report**		
15. SUBJECT TERMS		

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**	**Same as Report (SAR)**	**14**	

$S_{\mu_i}^{(1)}(\mathbf{x})$ (first-order) score function of mean, $S_{\mu_i}^{(1)}(\mathbf{x}) \equiv \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x})}{\partial \mu_i}$

$S_{\sigma_i}^{(1)}(\mathbf{x})$ (first-order) score function of STDEV, $S_{\sigma_i}^{(1)}(\mathbf{x}) \equiv \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x})}{\partial \sigma_i}$

1. INTRODUCTION

Reliability-based design optimization (RBDO) has been developed using the mean of the input random variable as the design variable. At the same time, the standard deviation (STDEV) of the input random variable is invariant in the conventional RBDO process. Many problems have been studied and solved assuming the STDEV is fixed in the RBDO process [1-10]. However, the fixed STDEV of an input random variable may not represent physical design problems appropriately. In the physical design problems, the tolerance of input random variables is given. The tolerance indicates the range within which the input random variable can vary, and it is usually a fixed percentage of the mean of the input random variable, which is the design variable in RBDO. Therefore, the range changes as the design variable changes during the RBDO process according to the percentage. The range can be interpreted as the STDEV of the input random variable; hence, STDEV should be considered to change during the RBDO process. In this case, the coefficient of variation (COV), which is the ratio of STDEV to mean, should be fixed in the RBDO process to represent the given tolerance.

Fixing the COV implies that the STDEV of the input random variable is linearly changing as the design variable changes; and a varying STDEV indicates that the uncertainty of the input random variable is changing as design iteration proceeds. If the uncertainty of the input random variable changes in each design iteration, the RBDO optimum design could become a moving target problem unless accurate design sensitivity of probability of failure, which is the probabilistic constraints in RBDO, is provided. In sensitivity-based RBDO, such as the first-order reliability method [1, 2, 11, 12], the second-order reliability method [13, 14], and the dimension reduction method [3, 15], the most probable point (MPP) needs to be found. However, the accurate position of MPP, where the design sensitivity is calculated [16], cannot be easily obtained because the position of the MPP would change significantly compared to the case when the STDEV is fixed. The reason for this is that the transformation from X-space (actual design variable space) to U-space (independent standard normal space), which is necessary to find the MPP, is affected by both the mean and the STDEV, which is being changed. Hence, an approximated RBDO process such as the Sequential Optimization and Reliability Assessment (SORA) method [6], which does not find an accurate MPP point in the design process, would suffer difficulty. That is why a specific method was developed for RBDO with fixed COV using SORA [17]. The sampling-based method does not use MPP; therefore, the same problem would not be encountered in sampling-based RBDO. However, the design sensitivity method for sampling-based RBDO with fixed COV has not been developed yet.

In this paper, the design sensitivity of probability of failure for sampling-based RBDO with fixed COV is developed. First, the sampling-based RBDO method is briefly revisited, and then the score functions of the mean and the STDEV of the input random variable are derived. In addition to marginal PDF, copulas [18, 19] are used in the derivation of score function to consider correlated input random variables. Next, the score functions are used to calculate design sensitivity for RBDO with fixed COV. Finally, the proposed design sensitivity is verified using numerical examples of RBDO with fixed COV.

2. REVIEW OF SAMPLING-BASED RBDO

A general RBDO problem is expressed as

$$\begin{aligned} &\text{minimize} \quad cost(\mathbf{d}) \\ &\text{subject to} \quad \begin{array}{l} P\left[G_j(\mathbf{X}) > 0\right] \le P_{F_j}^{\text{tar}}, \ \ j = 1, \dots, NC \\ \mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U, \mathbf{d} \in \mathbb{R}^{NDV}, \text{and} \ \ \mathbf{X} \in \mathbb{R}^{N} \end{array} \end{aligned} \tag{1}$$

where $\mathbf{X}$ is the N-dimensional random variable vector, $\mathbf{d}$ is the NDV-dimensional design variable vector, $\mathbf{d}^L$ is the lower design bound, $\mathbf{d}^U$ is the upper design bound, $P_{F_j}^{\text{tar}}$ is the target probability of failure for the j-th performance measure $G_j(\mathbf{X})$, and NC is the number of constraints. In sampling-based RBDO, the probabilistic constraint – probability of failure – is estimated using the Monte Carlo simulation (MCS) method as [4, 5, 20]

$$\begin{aligned} P_{F_j} \equiv P\left[G_j(\mathbf{X}) > 0\right] &= \int_{\Omega_{\mathbf{X}}} I_{\Omega_{F_j}}(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x} \\ &\cong \frac{1}{nMCS} \sum_{q=1}^{nMCS} I_{\Omega_{F_j}}[\mathbf{x}^{(q)}] \end{aligned} \tag{2}$$

where $nMCS$ is the number of MCS samples, $\mathbf{x}^{(q)}$ is the q-th realization (MCS sample) of $\mathbf{X}$, a failure set Ω_{F_j} is defined as $\Omega_{F_j} \equiv \{\mathbf{x}: G_j(\mathbf{x}) > 0\}$, and the indicator function is defined as

$$I_{\Omega_{F_j}}(\mathbf{x}) \equiv \begin{cases} 1, & \mathbf{x} \in \Omega_{F_j}, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

In Eq. (2), it can be seen that sampling-based RBDO does not require the sensitivity (gradient) of the performance measure $G_j(\mathbf{X})$ to evaluate the probability of failure. This is a significant merit for engineering problems in which it is difficult to obtain the gradient (e.g., explosion analysis) because RBDO can be accomplished without the gradient information using sampling-based RBDO. On the other hand, demerit of sampling-based RBDO is that the accurate estimation of probability of failure depends on the number of MCS samples $nMCS$. The existence of many $nMCS$ indicates accurate estimation; however, it also requires many evaluations of the performance measure $G_j(\mathbf{X})$, which could be computationally expensive. Hence, a surrogate model is used to calculate the probability of failure in Eq. (2) [4,

DISTRIBUTION STATEMENT A. Approved for public release; distribution is unlimited.

5]. In this paper, the surrogate model of performance measure is assumed to be available.

The probability of failure expression in Eq. (2) requires the input joint PDF of $f_{\mathbf{X}}(\mathbf{x})$. Also, MCS samples to calculate the probability of failure should be generated according to the input joint PDF. In this paper, the input joint PDF is defined using the marginal PDF of each input random variable and copula for each correlated pair of input random variables. Copula can connect two or more marginal PDFs of random variables to represent statistical correlation between the random variables and can generate a joint PDF using the copula density function. In this paper, statistical correlation between two random variables is considered. When the *k*-th correlated random variable pair X_i and X_j are correlated with the copula density function c_k, the joint PDF of X_i and X_j can be represented as [18, 19, 23]

$$\begin{aligned} &f_{X_iX_j}(x_i,x_j) \\ &= f_{X_i}(x_i;a_i,b_i)\, f_{X_j}(x_j;a_j,b_j)\, c_k(u,v;\theta_k) \end{aligned} \tag{4}$$

where $f_{X_i}(x_i;a_i,b_i)$ is the marginal PDF of the *i*-th input random variable X_i, a_i and b_i are two parameters of the marginal distribution, $c_k(u,v;\theta_k)$ is the copula density function of the *k*-th correlated pair, u and v are the cumulative distribution function (CDF) values of the correlated pair, and θ_k is the correlation coefficient for the copula. In this paper, it is assumed that each random variable has marginal distribution with two parameters, as is usually assumed in the RBDO process. Using the relationship in Eq. (4), the input joint PDF $f_{\mathbf{X}}(\mathbf{x})$ can be expressed as [18, 19, 23]

$$f_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^{N} f_{X_i}(x_i;a_i,b_i) \prod_{j=1}^{M} c_j(u,v;\theta_j) \tag{5}$$

where *N* is the number of random variables, and *M* is the number of correlated pairs. It is noted that the mean μ_k and the STDEV σ_k of the *k*-th random variable X_k are only related to the marginal PDF $f_{X_k}(x_k;a_k,b_k)$ and copula density function $c_m(u,v;\theta_m)$ when X_k is in the *m*-th correlated pair. The input joint PDF defined in Eq. (5) can be used to calculate the probability of failure in Eq. (2) and to obtain the design sensitivity of the probability of failure in the following section.

3. DESIGN SENSITIVITY AND SCORE FUNCTIONS

For an effective optimization process, an accurate and efficient design sensitivity needs to be provided, especially for RBDO with fixed COV. The design sensitivity of the probability of failure for sampling-based RBDO can be obtained using the score function. The score function of a variable is a partial derivative of the log function (log-derivative) of the input joint PDF in Eq. (5) with respect to the variable. The score function to estimate the stochastic sensitivity of the general functions of independent random variables was developed first [21]. The method was applied to functions of correlated Gaussian random variables as well [22]. In addition, a score function method to calculate the design sensitivity of the probability of failure has been developed for sampling-based RBDO with fixed STDEV for correlated input random variables [4, 5]. In this paper, the design sensitivity using the score function is further expanded to apply to sampling-based RBDO for correlated input random variables with fixed COV.

The design sensitivity of the probability of failure can be obtained by taking the derivative of the probability of failure expression in Eq. (2) with respect to a design variable d_i as

$$\begin{aligned} \frac{\partial}{\partial d_i} P_{F_j} &= \frac{\partial}{\partial d_i} \int_{\Omega_{\mathbf{X}}} I_{\Omega_{F_j}}(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x} \\ &= \int_{\Omega_{\mathbf{X}}} I_{\Omega_{F_j}}(\mathbf{x}) \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x})}{\partial d_i} f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}. \end{aligned} \tag{6}$$

In Eq. (6), it can be seen that a new term, the log-derivative of the input joint PDF $f_{\mathbf{X}}(\mathbf{x})$ with respect to the design variable d_i, is present in addition to the probability of failure expression in Eq. (2). This additional term is the first-order score function of the design variable d_i as

$$S_{d_i}^{(1)}(\mathbf{x}) \equiv \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x})}{\partial d_i}. \tag{7}$$

It is noted that the first-order score function in Eq. (7) does not include the sensitivity (gradient) of the performance measure $G_j(\mathbf{X})$. Therefore, the design sensitivity of the probability of failure in Eq. (6) can be calculated without the gradient of $G_j(\mathbf{X})$. Using the MCS method and the first-order score function, the design sensitivity in Eq. (6) can be estimated similar to Eq. (2) as [4, 5, 20]

$$\begin{aligned} \frac{\partial}{\partial d_i} P_{F_j} &= \int_{\Omega_{\mathbf{X}}} I_{\Omega_{F_j}}(\mathbf{x})\, S_{d_i}^{(1)}(\mathbf{x})\, f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x} \\ &\cong \frac{1}{nMCS} \sum_{q=1}^{nMCS} I_{\Omega_{F_j}}[\mathbf{x}^{(q)}]\, S_{d_i}^{(1)}[\mathbf{x}^{(q)}]. \end{aligned} \tag{8}$$

Let the design variable d_i be μ_k, which is the mean of the *k*-th random variable X_k. Also, let σ_k and COV_k be the STDEV and COV of the random variable X_k. Then the first-order score function in Eq. (7) can be further derived as

$$\begin{aligned} &S_{d_i}^{(1)}(\mathbf{x}) \\ &= \begin{cases} S_{\mu_k}^{(1)}(\mathbf{x}) & \text{if } \sigma_k \text{ is fixed,} \\ S_{\mu_k}^{(1)}(\mathbf{x}) + COV_k\, S_{\sigma_k}^{(1)}(\mathbf{x}) & \text{if: } COV_k \text{ is fixed.} \end{cases} \end{aligned} \tag{9}$$

where $S_{\mu_k}^{(1)}(\mathbf{x}) \equiv \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x})}{\partial \mu_k}$ and $S_{\sigma_k}^{(1)}(\mathbf{x}) \equiv \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x})}{\partial \sigma_k}$ are first-order score functions of mean and STDEV, respectively. Hence, to evaluate the design sensitivity in Eq. (6), the score-functions $S_{\mu_k}^{(1)}(\mathbf{x})$ and $S_{\sigma_k}^{(1)}(\mathbf{x})$ are required. As explained earlier, μ_k and

σ_k are related only to marginal PDF $f_{X_k}(x_k; a_k, b_k)$ and copula density function $c_m(u, v, ; \theta_m)$ in the joint PDF $f_{\mathbf{X}}(\mathbf{x})$ in Eq. (5). Therefore, the score functions $S_{\mu_k}^{(1)}(\mathbf{x})$ and $S_{\sigma_k}^{(1)}(\mathbf{x})$ can be obtained as

$$\begin{aligned} S_{\mu_k}^{(1)}(\mathbf{x}) &= \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x})}{\partial \mu_k} \\ &= \frac{\partial \ln f_{X_k}(x_k; a_k, b_k)}{\partial \mu_k} + \frac{\partial \ln c_m(u, v; \theta_m)}{\partial \mu_k} \end{aligned} \tag{10}$$

and

$$\begin{aligned} S_{\sigma_k}^{(1)}(\mathbf{x}) &= \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x})}{\partial \sigma_k} \\ &= \frac{\partial \ln f_{X_k}(x_k; a_k, b_k)}{\partial \sigma_k} + \frac{\partial \ln c_m(u, v; \theta_m)}{\partial \sigma_k}. \end{aligned} \tag{11}$$

In Eqs. (8) and (9), it can be seen that log-derivatives of $f_{X_k}(x_k; a_k, b_k)$ and $c_m(u, v, ; \theta_m)$ with respect to μ_k and σ_k have to be derived. Here, it is noted that the log-derivative of the copula density function can be ignored when the input random variable X_k is statistically independent.

3.1 Log-derivatives of marginal PDF with respect to mean and STDEV

Seven widely used marginal distribution types are considered in this paper. Their PDFs and CDFs are shown in Appendix, Table A.1. It can be seen that they have two parameters a and b in their analytical forms. The two parameters can uniquely determine the mean and the STDEV of the input random variable and vice versa, as shown in Appendix, Table A.2. Therefore, the log-derivatives of the marginal PDF in Eqs. (10) and (11) can be represented using the two parameters a and b as

$$\frac{\partial \ln f_{X_k}(x_k; a_k, b_k)}{\partial \mu_k} = \frac{\partial \ln f_{X_k}}{\partial a_k}\frac{\partial a_k}{\partial \mu_k} + \frac{\partial \ln f_{X_k}}{\partial b_k}\frac{\partial b_k}{\partial \mu_k} \tag{12}$$

and

$$\frac{\partial \ln f_{X_k}(x_k; a_k, b_k)}{\partial \sigma_k} = \frac{\partial \ln f_{X_k}}{\partial a_k}\frac{\partial a_k}{\partial \sigma_k} + \frac{\partial \ln f_{X_k}}{\partial b_k}\frac{\partial b_k}{\partial \sigma_k}. \tag{13}$$

where a_k and b_k are the two parameters of the input random variable X_k. The right sides of Eqs. (12) and (13) are composed of the log-derivatives of the marginal PDF with respect to the two parameters and their derivatives to the mean and STDEV. Using the analytical expression of the marginal PDF in Table A.1, the log-derivatives of the marginal PDF with respect to the two parameters a and b are obtained as shown in Table 1. Using the relationship in Table A.2, derivatives of a and b with respect to the mean μ and the STDEV σ are also obtained as shown in Table 2. By substituting the derivative terms in Tables 1 and 2 for Eqs. (12) and (13), the log-derivatives of the marginal PDF with respect to the mean and STDEV in Eqs. (10) and (11) can be obtained.

Table 1 Log-derivatives of Marginal PDF with Respect to Parameters

Distribution type	$\dfrac{\partial \ln f_{X_i}}{\partial a_i}$	$\dfrac{\partial \ln f_{X_i}}{\partial b_i}$
Normal	$\dfrac{x_i - a_i}{b_i^2}$	$-\dfrac{1}{b_i} + \dfrac{(x_i - a_i)^2}{b_i^3}$
Lognormal	$\dfrac{\ln x_i - a_i}{b_i^2}$	$-\dfrac{1}{b_i} + \dfrac{(\ln x_i - a_i)^2}{b_i^3}$
Weibull	$\dfrac{b_i}{a_i}\left[-1 + \left(\dfrac{x_i}{a_i}\right)^{b_i}\right]$	$\dfrac{1}{b_i} + \left[1 - \left(\dfrac{x_i}{a_i}\right)^{b_i}\right](\ln x_i - \ln a_i)$
Gumbel	$-\dfrac{1}{b_i} + \dfrac{x_i - a_i}{b_i^2}\left[1 - \exp\left(-\dfrac{x_i - a_i}{b_i}\right)\right]$	$\dfrac{1}{b_i}\left[1 - \exp\left(-\dfrac{x_i - a_i}{b_i}\right)\right]$
Gamma	$\ln x_i - \psi(a_i) - \ln b_i$ *	$\dfrac{x_i}{b_i^2} - \dfrac{a_i}{b_i}$
Extreme	$\dfrac{1}{b_i}\left[\exp\left(\dfrac{x_i - a_i}{b_i}\right) - 1\right]$	$\dfrac{x_i - a_i}{b_i^2}\left[\exp\left(\dfrac{x_i - a_i}{b_i}\right) - 1\right] - \dfrac{1}{b_i}$
Extreme Type II	$\dfrac{1}{a_i} + (\ln b_i - \ln x_i)\left[1 - \left(\dfrac{b_i}{x_i}\right)^{a_i}\right]$	$\dfrac{a_i}{b_i}\left[1 - \left(\dfrac{b_i}{x_i}\right)^{a_i}\right]$

* $\psi(\bullet)$: Digamma function, $\psi(s) \equiv \Gamma'(s)/\Gamma(s)$

Table 2 Derivatives of Parameters with Respect to Mean and STDEV

Distribution type	$\frac{\partial a_i}{\partial \mu_i}$	$\frac{\partial b_i}{\partial \mu_i}$	$\frac{\partial a_i}{\partial \sigma_i}$	$\frac{\partial b_i}{\partial \sigma_i}$
Normal	1	0	0	1
Lognormal	$\frac{1}{\mu_i}\left(1+\frac{\sigma_i^2}{\mu_i^2+\sigma_i^2}\right)$	$-\frac{\sigma_i^2}{b_i\,\mu_i\,(\mu_i^2+\sigma_i^2)}$	$-\frac{\sigma_i}{\mu_i^2+\sigma_i^2}$	$\frac{\sigma_i}{b_i\,(\mu_i^2+\sigma_i^2)}$
Weibull	$\begin{bmatrix}\partial a_i/\partial \mu_i \\ \partial b_i/\partial \mu_i\end{bmatrix} = \mathbf{A}^{-1}\begin{bmatrix}1 \\ \mu_i\end{bmatrix}$		$\begin{bmatrix}\partial a_i/\partial \sigma_i \\ \partial b_i/\partial \sigma_i\end{bmatrix} = \mathbf{A}^{-1}\begin{bmatrix}0 \\ \sigma_i\end{bmatrix}$	
	where $\mathbf{A} = \begin{bmatrix}\Gamma(1+1/b_i) & -\frac{1}{b_i^2}\,\Gamma(1+1/b_i)\,\psi(1+1/b_i) \\ a_i\,\Gamma(1+2/b_i) & -\frac{a_i^2}{b_i^2}\,\Gamma(1+2/b_i)\,\psi(1+2/b_i)\end{bmatrix}$			
Gumbel	1	0	$-3.4632\,\frac{\sigma_i}{\pi^2 b_i}$	$\frac{6\sigma_i}{\pi^2 b_i}$
Gamma	$\frac{2}{b_i}$	$-\frac{1}{a_i}$	$-\frac{2\sqrt{a_i}}{b_i}$	$\frac{2}{\sqrt{a_i}}$
Extreme	1	0	$\frac{0.5772\sqrt{6}}{\pi}$	$\frac{\sqrt{6}}{\pi}$
Extreme Type II	$\begin{bmatrix}\partial a_i/\partial \mu_i \\ \partial b_i/\partial \mu_i\end{bmatrix} = \mathbf{B}^{-1}\begin{bmatrix}1 \\ \mu_i\end{bmatrix}$		$\begin{bmatrix}\partial a_i/\partial \sigma_i \\ \partial b_i/\partial \sigma_i\end{bmatrix} = \mathbf{B}^{-1}\begin{bmatrix}0 \\ \sigma_i\end{bmatrix}$	
	where $\mathbf{B} = \begin{bmatrix}\frac{b_i}{a_i^2}\,\Gamma(1-1/a_i)\,\psi(1-1/a_i) & \Gamma(1-1/a_i) \\ \frac{b_i^2}{a_i^2}\,\Gamma(1-2/a_i)\,\psi(1-2/a_i) & b_i\,\Gamma(1-2/a_i)\end{bmatrix}$			

3.2 Log-derivatives of copula density function with respect to mean and STDEV

The second terms in the right sides of Eqs. (10) and (11) are the log-derivatives of the copula density function. As explained earlier, the log-derivatives of the copula density function can be ignored when X_k is an independent random variable because the copula density function for independent random variables is one and its log-derivative is zero. If X_k is statistically correlated and $u = F_{X_k}(x_k; a_k, b_k)$ is the CDF value of x_k, the log-derivatives of the copula density function become

$$\frac{\partial \ln c_m(u, v; \theta_m)}{\partial \mu_k} = \frac{\partial \ln c_m}{\partial u}\frac{\partial u}{\partial \mu_k} \tag{14}$$

and

$$\frac{\partial \ln c_m(u, v; \theta_m)}{\partial \sigma_k} = \frac{\partial \ln c_m}{\partial u}\frac{\partial u}{\partial \sigma_k}. \tag{15}$$

For various copula types, the log-derivatives of the copula density function with respect to u $(\partial \ln c_m / \partial u)$, in the right sides of Eqs. (14) and (15), are well summarized in Ref. 4. The other terms $\partial u / \partial \mu_k$ and $\partial u / \partial \sigma_k$ are obtained in this paper. In Table A.1, it can be seen that the marginal CDF of the input random variable X_k, $u = F_{X_k}(x_k; a_k, b_k)$ is also a function of the two parameters a_k and b_k. Therefore, the derivatives of CDF u can be expressed using the two parameters as

$$\frac{\partial u}{\partial \mu_k} = \frac{\partial u}{\partial a_k}\frac{\partial a_k}{\partial \mu_k} + \frac{\partial u}{\partial b_k}\frac{\partial b_k}{\partial \mu_k} \tag{16}$$

and

$$\frac{\partial u}{\partial \sigma_k} = \frac{\partial u}{\partial a_k}\frac{\partial a_k}{\partial \sigma_k} + \frac{\partial u}{\partial b_k}\frac{\partial b_k}{\partial \sigma_k}. \tag{17}$$

In Eqs. (16) and (17), the derivative of u with respect to the two parameters a_k and b_k are necessary. Using the analytical representations of the marginal CDFs in Table A.1, the derivatives of the seven marginal CDFs are derived and summarized in Table 3. Using Ref. 4 and Table 3, the log-derivatives of the copula density function in Eqs. (14) and (15) are available. Then, all terms for Eq. (9) are obtained, and the design sensitivity of the probability of failure for sampling-based RBDO with fixed COV can be calculated using Eq. (8).

3.3 Score-functions for sampling-based RBDO with fixed COV

Using the derivatives obtained in previous sections, the design sensitivity of probability of failure for sampling-based RBDO with fixed COV can be obtained. Unlike the case in

DISTRIBUTION STATEMENT A. Approved for public release; distribution is unlimited.

Table 3 Derivative of CDF with Respect to Parameters

Distribution type	$\dfrac{\partial u}{\partial a_i}$	$\dfrac{\partial u}{\partial b_i}$
Normal	$-\dfrac{1}{b_i}\phi\left(\dfrac{x_i-a_i}{b_i}\right)$	$-\dfrac{x_i-a_i}{b_i^2}\phi\left(\dfrac{x_i-a_i}{b_i}\right)$
Lognormal	$-\dfrac{1}{b_i}\phi\left(\dfrac{\ln x_i-a_i}{b_i}\right)$	$-\dfrac{\ln x_i-a_i}{b_i^2}\phi\left(\dfrac{\ln x_i-a_i}{b_i}\right)$
Weibull	$-\dfrac{b_i}{a_i}\left(\dfrac{x_i}{a_i}\right)^{b_i}(1-u)$	$(\ln x_i-\ln a_i)\left(\dfrac{x_i}{a_i}\right)^{b_i}(1-u)$
Gumbel	$-\dfrac{1}{b_i}\exp\left(-\dfrac{x_i-a_i}{b_i}\right)u$	$-\dfrac{x_i-a_i}{b_i^2}\exp\left(-\dfrac{x_i-a_i}{b_i}\right)u$
Gamma	$\Gamma(a_i)\psi(a_i)-\ln x_i\,\Gamma(a_i,x_i/b_i) - x_iT(3,a_i,x_i/b_i)^{\dagger}$	$-\dfrac{e^{-x_i/b_i}(x_i/b_i)^{a_i}}{b_i\Gamma(a_i)}$
Extreme	$\dfrac{1}{b_i}\exp\left(\dfrac{x_i-a_i}{b_i}\right)(u-1)$	$\dfrac{x_i-a_i}{b_i^2}\exp\left(\dfrac{x_i-a_i}{b_i}\right)(u-1)$
Extreme Type II	$-(\ln b_i-\ln x_i)\left(\dfrac{b_i}{x_i}\right)^{a_i}u$	$-\dfrac{a_i}{b_i}\left(\dfrac{b_i}{x_i}\right)^{a_i}u$

which STDEV is fixed, the design sensitivity for RBDO with fixed COV requires all terms in Eqs. (10) to (17) for its calculation. Therefore, the design sensitivity expression for RBDO with fixed COV would be more involved. However, for some marginal distribution types, the expression could be simpler.

The second expression on the right side of Eq. (9) is the first-order score function of design variable d_i for RBDO with fixed COV. When Eqs. (10) to (17) are substituted for the expression, the score function becomes

$$S_{d_i}^{(1)}(\mathbf{x}) = \left(\frac{\partial \ln f_{X_k}}{\partial a_k}+\frac{\partial \ln c_m}{\partial u}\frac{\partial u}{\partial a_k}\right)\left(\frac{\partial a_k}{\partial \mu_k}+COV_k\frac{\partial a_k}{\partial \sigma_k}\right) + \left(\frac{\partial \ln f_{X_k}}{\partial b_k}+\frac{\partial \ln c_m}{\partial u}\frac{\partial u}{\partial b_k}\right)\left(\frac{\partial b_k}{\partial \mu_k}+COV_k\frac{\partial b_k}{\partial \sigma_k}\right). \tag{18}$$

In Table 2, the derivatives of the two parameters a and b with respect to mean and STDEV have been shown. It can be found that the following relationship

$$\frac{\partial b_k}{\partial \mu_k}+COV_k\frac{\partial b_k}{\partial \sigma_k}=0 \tag{19}$$

holds for the Lognormal distribution. Therefore, Eq. (18) for the Lognormal distribution becomes a simpler form as

$$S_{d_i}^{(1)}(\mathbf{x}) = \left(\frac{\partial \ln f_{X_k}}{\partial a_k}+\frac{\partial \ln c_m}{\partial u}\frac{\partial u}{\partial a_k}\right)\left(\frac{\partial a_k}{\partial \mu_k}+COV_k\frac{\partial a_k}{\partial \sigma_k}\right). \tag{20}$$

Also, for the Weibull distribution, Eq. (19) holds; furthermore, the relationship in Eq. (21), which can be derived using the derivatives in Table 2, holds as well.

$$\frac{\partial a_k}{\partial \mu_k}+COV_k\frac{\partial a_k}{\partial \sigma_k}=\frac{1}{\Gamma(1+1/b_k)} \tag{21}$$

Hence, the score function in Eq. (18) can be expressed in a simpler form for the Weibull distribution as

$$S_{d_i}^{(1)}(\mathbf{x}) = \left(\frac{\partial \ln f_{X_k}}{\partial a_k}+\frac{\partial \ln c_m}{\partial u}\frac{\partial u}{\partial a_k}\right)\frac{1}{\Gamma(1+1/b_k)}. \tag{22}$$

The Extreme type-II distribution also has special relationships when COV is fixed as

$$\frac{\partial a_k}{\partial \mu_k}+COV_k\frac{\partial a_k}{\partial \sigma_k}=0 \tag{23}$$

and

$$\frac{\partial b_k}{\partial \mu_k}+COV_k\frac{\partial b_k}{\partial \sigma_k}=\frac{1}{\Gamma(1-1/a_k)}. \tag{24}$$

† $\Gamma(\bullet,\bullet)$: Upper incomplete gamma function, $\Gamma(s,x)\equiv\int_x^\infty t^{s-1}e^{-t}\,dt$,
$T(\bullet,\bullet,\bullet)$: A case of Meijer G-function. See Ref. 24.

Finally, the Extreme type-II distribution has a simpler form of Eq. (18) as

$$S_{d_i}^{(1)}(\mathbf{x}) = \left(\frac{\partial \ln f_{X_k}}{\partial b_k} + \frac{\partial \ln c_m}{\partial u}\frac{\partial u}{\partial b_k}\right)\frac{1}{\Gamma(1-1/a_k)}. \quad (25)$$

Because the score function $S_{d_i}^{(1)}(\mathbf{x})$ is derived analytically, the design sensitivity of probability of failure in Eq. (8) can be calculated along with calculation of the probability of failure in Eq. (2) with small additional effort. Hence, the design sensitivity for RBDO with fixed COV can be evaluated efficiently using the score functions during the calculation of the probability of failure.

4. NUMERICAL EXAMPLES

The effectiveness and efficiency of the developed method for the design sensitivity of the probability of failure are tested using numerical examples in this section. The developed design sensitivity is compared to the sensitivity using the finite difference method (FDM) to verify its accuracy. The FDM results are obtained after a number of trial-and error process in the determination of appropriate step size to yield accurate design sensitivity. Then, using the developed design sensitivity, RBDO with fixed COV is performed for a 2-D mathematical problem and a 14-D engineering problem to understand how well the developed design sensitivity method works for the RBDO process.

4.1 Accuracy check

The accuracy of the developed design sensitivity of probability of failure is verified using a highly nonlinear performance measure

$$\begin{aligned} G(\mathbf{X}) = 0.7361 + (Y-6)^2 + (Y-6)^3 \\ -0.6 \times (Y-6)^4 + Z \end{aligned} \quad (26)$$

where

$$\begin{bmatrix} Y \\ Z \end{bmatrix} = \begin{bmatrix} 0.9063 & 0.4226 \\ 0.4226 & -0.9063 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

and $G(\mathbf{X}) > 0$ indicates failure. For thorough testing, five different input models, which include all marginal distribution and copula types dealt with in the previous sections are used for the testing. The design variables d_1 and d_2 are the means of the input random variables X_1 and X_2, respectively. In the design sensitivity calculation, the COVs of both random variables are fixed.

The accuracy of the developed design sensitivity of the probability of failure is verified by comparing it with the forward FDM design sensitivity. The result of the developed design sensitivity and the FDM design sensitivity is summarized in Table 5. The ratio of the developed sensitivity to the FDM sensitivity is listed in the 'Accuracy' column of the table. The ratio varies from 97.1% to 102.6%, which indicates that the developed sensitivity agrees with the FDM design sensitivity. Therefore, it can be concluded that the developed design sensitivity is accurate.

Table 4 Input Models for Accuracy Test

	X_1				X_2				Copula	
	Type	Mean	STDEV	COV	Type	Mean	STDEV	COV	Type	Tau
A	Weibull	4	0.5	0.125	Normal	2.5	0.5	0.2	Clayton	0.3
B	Gumbel				Lognormal				Frank	0.3
C	Gamma				Extreme				FGM	0.2
D	Extreme				Normal				Gaussian	0.3
E	Extreme II				Lognormal				AMH	0.33

Table 5 Comparison of FDM and Developed Design Sensitivity

Input model	d_1				d_2			
	Perturb	FDM	Developed	Accuracy	Perturb	FDM	Developed	Accuracy
A	0.1%	0.600	0.597	99.5%	0.1%	−0.135	−0.133	98.5%
B	0.1%	0.346	0.338	97.7%	1.0%	−0.107	−0.106	99.1%
C	0.1%	0.458	0.469	102.4%	1.0%	−0.174	−0.169	97.1%
D	0.1%	0.639	0.638	99.8%	0.5%	−0.144	−0.147	102.1%
E	0.2%	0.309	0.306	99.0%	0.5%	−0.077	−0.079	102.6%

DISTRIBUTION STATEMENT A. Approved for public release; distribution is unlimited.

In this example, the FDM design sensitivity of the probability of failure is used as a reference value to check the accuracy of the developed design sensitivity. To obtain the accurate FDM design sensitivity, two conditions − appropriate perturbation size and highly accurate probability of failure at both perturbed and original design − should be satisfied. However, obtaining appropriate perturbation size is not an easy task. A large perturbation size cannot provide accurate design sensitivity of a nonlinear function, while a small perturbation size can suffer numerical noise. In this example, perturbation sizes of 0.1%, 0.2%, 0.5% and 1% of the design variable value are used, as shown in Table 5. For different cases and different design variables, perturbation size should be carefully chosen; hence, several perturbation sizes are tested, and the perturbation size that shows a stable design sensitivity value is selected. An appropriate perturbation size at one design configuration may not be applicable at the other design configuration because the nonlinearity of the probability of failure changes as the design changes. Hence, it is not easy to use the FDM design sensitivity in the optimization process. While the developed design sensitivity method does not depend on the perturbation size, it shows the same accuracy as the FDM design sensitivity; hence, it is more suitable for the optimization.

The second condition – accurate probability of failure – is also important in order to obtain the correct design sensitivity of the probability of failure using the FDM. In Table 5, the FDM design sensitivity to d_1 in input model A is 0.600. To obtain this value, the finite difference 0.00240 between two probabilities of failure, 0.13726 at the perturbed design and 0.13486 at the original design, is divided by the perturbation size 0.004, which is 0.1% of d_1. It is noted that the significant digit in the two probabilities of failure should be larger than five to obtain three significant digits in the finite difference value. If an estimated value of probability of failure is needed, three significant digits might be enough. However, because the finite difference value is needed to calculate the FDM design sensitivity, more accurate probabilities of failure are required to have enough significant digits in the finite difference value calculation. For this reason, in this example, 200 million MCS samples were required to accurately calculate the probabilities of failure at the original and perturbed design; hence, a total of 400 million samples are used to calculate one value of FDM design sensitivity. On the other hand, the developed method evaluates the design sensitivity without calculating the finite difference value. Hence, we can use much smaller number of MCS samples to calculate an accurate design sensitivity. In this example, one million MCS samples, which are 0.25% of the samples for the FDM design sensitivity, are used. Hence, it can be seen that the developed design sensitivity is efficient and suitable for the optimization process.

4.2 2-D mathematical example

To find how well the developed design sensitivity method works in the optimization process, RBDO is performed for a 2-D mathematical example. The RBDO formulation is shown as

$$\begin{aligned} &\textit{minimize} \quad C(\mathbf{d}) = -\frac{(d_1+d_2-10)^2}{30} - \frac{(d_1-d_2+10)^2}{120} \\ &\text{subject to} \quad P[G_i(\mathbf{X})>0] \le p_{F_i}^{Tar} = 2.275\%, \\ &\qquad i=1,\dots,3, \quad \mathbf{d}^L < \mathbf{d} < \mathbf{d}^U, \\ &\qquad \mathbf{d}\in\mathbb{R}^2 \ \ and \ \ \mathbf{X}\in\mathbb{R}^2 \end{aligned} \tag{27}$$

where $\mathbf{X} = [X_1, X_2]^T$, $\mathbf{d} = [d_1, d_2]^T$, $\mathbf{d}^L = [0, 0]^T$, $\mathbf{d}^U = [10, 10]^T$, and the three constraints are given by

$$\begin{aligned} G_1(\mathbf{X}) &= 1 - \frac{X_1^2 X_2}{20}, \\ G_2(\mathbf{X}) &= -1 + (0.9063X_1 + 0.4226X_2 - 6)^2 \\ &\quad +(0.9063X_1 + 0.4226X_2 - 6)^3 \\ &\quad -0.6(0.9063X_1 + 0.4226X_2 - 6)^4 \\ &\quad -(-0.4226X_1 + 0.9063X_2), \\ G_3(\mathbf{X}) &= 1 - \frac{80}{X_1^2 + 8X_2 + 5}. \end{aligned} \tag{28}$$

The contours of the cost function $C(\mathbf{d})$ in Eq. (27) and limit states of the three constraints in Eq. (28) are shown in Fig. 1.

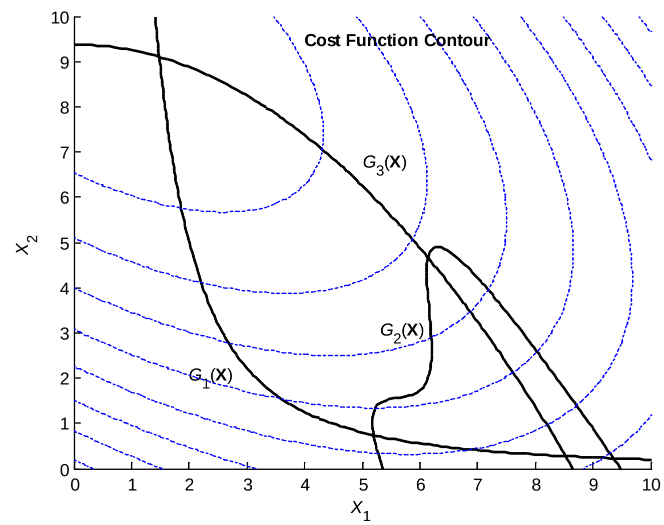


Figure 1 Cost Function and Limit State Contours for 2-D Mathematical Example

In this example, two different input models are used for RBDO as shown in Table 6. The two models have the same means, STDEVs, COVs, copula type, and Kendall's tau, while the marginal distribution types of X_1 and X_2 are different. The two design variables d_1 and d_2 of the example are the means of X_1 and X_2, respectively, and the COVs of X_1 and X_2 are fixed during the RBDO. To calculate the probabilities of failure and the design sensitivities for the probabilities of failure, one million MCS samples are used in this example.

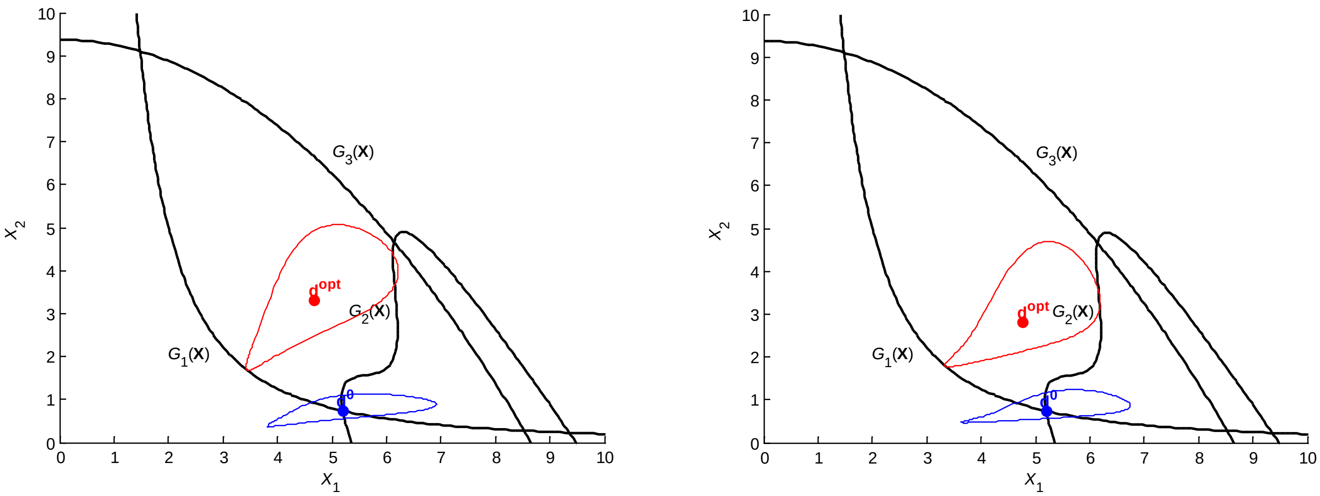
DISTRIBUTION STATEMENT A. Approved for public release; distribution is unlimited.

Table 6 Input Models for 2-D Mathematical Example

Input model	X_1				X_2				Copula	
	Type	Mean	STDEV	COV	Type	Mean	STDEV	COV	Type	Tau
A	Lognormal	5.19	0.7785	0.15	Normal	0.74	0.185	0.25	Clayton	0.5
B	Normal				Gumbel					

Table 7 RBDO Result of 2-D Mathematical Example

Input model	Optimum design				Iter.	Reli. Anal.	Cost	Probability of failure (%)		
	d_1	d_2	σ_1	σ_2				G_1	G_2	G_3
A	4.6650	3.3192	0.6998	0.8298	8	9	−1.2082	2.2864	2.2986	1.5631
B	4.7610	2.8072	0.7141	0.7018	9	10	−1.3879	2.2678	2.2955	0.7259



(a) Input model A *(b) Input model B*

Figure 2 2-σ Contour of Input Model at Initial Design and Optimum Design

The result of RBDO using the developed design sensitivity for RBDO with fixed COV is shown in Table 7. It can be seen that the STDEVs are changed as the design variables change. However, the COVs are kept during the optimization process to 0.15 for X_1 and 0.25 for X_2. Input model A requires eight design iterations and nine reliability analyses. This means that only one additional line search is needed in the optimization process, as an accurate design sensitivity is provided. Similarly, RBDO of input model B converges in nine design iterations with one additional line search, which indicates the effectiveness of the design sensitivity as well. In both cases, a 2.275% target probability of failure is satisfied under 1% relative convergence criteria for the probabilistic constraints. Therefore, both optimum designs satisfy the probabilistic constraints.

In Fig. 2, 2-σ contours of the input model at initial design $\mathbf{d}^0$ and RBDO optimum design $\mathbf{d}^{\text{opt}}$ are shown. As the limit state functions are not linear and some input random variables do not follow normal distribution, the 2-σ contours at RBDO optimum designs $\mathbf{d}^{\text{opt}}$ do not meet the limit states tangentially. Note that the area of the 2-σ contours are dramatically changed from the initial design to the optimum design because the STDEV of X_2 changes significantly as the second design variable d_2 changes. Especially for input model A, the 2-σ contour of the optimum design fits in the feasible region as shown in Fig. 2(a), so a slight movement of the design could violate one or more probabilistic

constraints. Hence, unless accurate design sensitivity is provided, optimum design will not be easy to obtain readily.

The same RBDO with fixed COV has been carried out once again using inaccurate design sensitivity by ignoring the second term $COV_k\, S_{\sigma_k}^{(1)}(\mathbf{x})$ in the second design sensitivity expression in Eq. (9). In other words, the effect of the change of STDEV is ignored. The RBDO result is shown in Table 8. In both cases, the optimization process does not converge to the optimum design as 2.275% target probability failure cannot be achieved. In addition, the number of reliability analyses is over 18, which is almost twice that of previous cases when an accurate design sensitivity of probability of failure was provided. Therefore, the STDEV effect in the design sensitivity calculation is necessary for appropriate and efficient convergence to the correct optimum design of RBDO with fixed COV.

4.3 14-D engineering example

An RBDO has been performed for a 14-D engineering problem. This example is a car noise, vibration, and harshness (NVH) and crash-safety problem. There are 11 constraints, which include full frontal impact, 40% offset frontal impact, and NVH, as shown in Table 9. Surrogate models for the constraints and cost function for RBDO has been provided by Ford Motor Company. Originally, this example had 44 input random variables, which means it was a 44-D RBDO problem. However, the number of random variables has been reduced to 14 according to a variable screening procedure [25].

Input random variables are the thickness of the plates in a car body. They have Normal, Lognormal, Weibull, and Gamma distributions as marginal distribution types. The design variables are the means of the input random variables, and the COVs are fixed to be constants. There are two correlated pairs of X_6 - X_7 and X_{25} - X_{26}, whose correlations are represented using the Clayton and Gaussian copulas, respectively. The detailed information of the input model is shown in Table 10; $\mathbf{d}^B$ indicates baseline design, which is the initial design of the optimization.

The RBDO formulation of the 14-D engineering example is shown in Eq. (28). Cost function is the weight of the car body and a function of design variable vector $\mathbf{d}$, not input random variable vector $\mathbf{X}$. Hence, the cost function is a deterministic function, while the constraint functions are a random function of the vector $\mathbf{X}$.

$$\begin{aligned} &minimize \quad C(\mathbf{d}) = \text{Weight}(\mathbf{d}) \\ &\text{subject to} \quad P[G_i(\mathbf{X}) > Baseline_i] \le p_{F_i}^{Tar} = 10\%, \\ &\qquad i = 1, \ldots, 11,\ \ \mathbf{d}^L < \mathbf{d} < \mathbf{d}^U, \\ &\qquad \mathbf{d} \in \mathbb{R}^{14}\ \ and\ \ \mathbf{X} \in \mathbb{R}^{14} \end{aligned} \tag{29}$$

Table 8 RBDO Result of 2-D Mathematical Example with Inaccurate Design Sensitivity

Input model	Optimum design				Iter.	Reli. Anal.	Cost	Probability of failure (%)		
	d_1	d_2	σ_1	σ_2				G_1	G_2	G_3
A	4.6310	2.8821	0.6946	0.7205	6	20	−1.3565	3.5811	2.2308	0.5294
B	4.7572	2.6993	0.7136	0.6748	7	18	−1.4272	2.6379	2.2408	0.5165

Table 9 Performance Measures of 14-D Engineering Example

Mode		Function	Value	Feasibility decision
Safety	Full frontal impact	G_1	Chest G	$\le Baseline_i$
		G_2	Crush displacement	
	40% offset impact	G_3	Brake pedal	
		G_4	Footrest	
		G_5	Left toepan	
		G_6	Center toepan	
		G_7	Right toepan	
		G_8	Left IP	
		G_9	Right IP	
NVH		G_{10}	Torsion mode	
		G_{11}	Vertical bending mode	

DISTRIBUTION STATEMENT A. Approved for public release; distribution is unlimited.

Table 10 Input Model for 14-D Engineering Example

RVs	Dist. type	Mean	COV	$\mathbf{d}^B$	$\mathbf{d}^L$	$\mathbf{d}^U$	Correlation
X_1	Normal	d_1	0.025	1.9	1.5	2.3	-
X_2	Normal	d_2	0.025	1.91	1.5	2.3	
X_3	Normal	d_3	0.025	2.51	2.0	3.0	
X_4	Normal	d_4	0.025	2.4	1.9	2.9	
X_5	Normal	d_5	0.025	2.55	2.0	3.1	
X_6	Normal	d_6	0.025	2.25	1.8	2.7	Clayton copula $\tau = 0.5$
X_7	Normal	d_7	0.025	2.25	1.8	2.7	
X_8	Gamma	d_8	0.02	1.5	1.2	1.8	-
X_{10}	Gamma	d_9	0.02	1.28	0.9	1.6	
X_{20}	Lognormal	d_{10}	0.025	1.22	0.9	1.5	
X_{23}	Lognormal	d_{11}	0.04	0.75	0.6	1.0	
X_{25}	Lognormal	d_{12}	0.04	0.65	0.5	0.8	Gaussian copula $\tau = 0.3$
X_{26}	Lognormal	d_{13}	0.04	0.85	0.6	1.1	
XN_1	Weibull	d_{14}	0.04	0.9	0.7	1.1	-

As shown in Table 9, there are 11 constraints in this example. They are safe if their values are less than the baseline values. The target probability of failure for all performance measures is 10%. Design variable vector **d** starts from the baseline design $\mathbf{d}^B$ and has to be in between lower bound $\mathbf{d}^L$ and upper bound $\mathbf{d}^U$. Component values of $\mathbf{d}^B$, $\mathbf{d}^L$, and $\mathbf{d}^U$ have been shown in Table 10. For the efficient optimization process, deterministic design optimization (DDO) is performed first. Generally, the DDO optimum design is close to the RBDO optimum design because the cost function is minimized and active constraints are determined in DDO. The RBDO is carried out to increase the reliability (reduce the probability of failure) of the active constraints while minimizing the cost. Hence, the RBDO optimum design could be obtained efficiently by starting RBDO from the DDO optimum design. The sampling-based RBDO has been carried out using 200,000 MCS samples. The optimization process is completed using 30 reliability analyses. Among the 30 analyses, 19 are design iterations, which means 11 are additional line searches. This could be a good indication that the provided design sensitivity is accurate and effective because the accurate design sensitivity of probability of failure reduces the total number of reliability analyses. If an inaccurate design sensitivity was used, the optimization would not find a better solution in the direction of the design sensitivity. In that case, the search for the next design point along the line search direction would come back to the current design point, and all the line searches would be wasted in that design iteration. Therefore, the entire optimization process becomes more efficient when an accurate design sensitivity is provided. As explained earlier, the design sensitivity calculation can be done during the reliability analysis (probability of failure estimation) with a little additional cost. Hence, the design sensitivity calculation itself is efficient as well. Therefore, the developed design sensitivity method would provide the utmost efficiency due to the reduction of line searches and its efficient calculation.

Table 11 Optimal Designs

Design variable	Initial design	DDO	RBDO
d_1	1.9	1.8768	1.8273
d_2	1.91	1.8857	2.1891
d_3	2.51	2.3008	2.8540
d_4	2.4	1.9753	1.9701
d_5	2.55	2.8003	2.7060
d_6	2.25	2.4358	2.2611
d_7	2.25	2.1066	2.3265
d_8	1.5	1.5353	1.7841
d_9	1.28	0.9	0.9
d_{10}	1.22	0.9	0.9
d_{11}	0.75	0.6	0.6
d_{12}	0.65	0.5522	0.5647
d_{13}	0.85	1.1	1.099
d_{14}	0.9	0.7	0.7

The obtained optimum designs are shown in Table 11. In the DDO procedure, $d_9 \sim d_{11}$ and d_{14} went to their lower bounds,

DISTRIBUTION STATEMENT A. Approved for public release; distribution is unlimited.

while d_{13} reached its upper bound; they remain on their bounds until RBDO converges. The other variables, d_1 ~ d_8 and d_{12}, have changed to find designs that are more reliable than the DDO optimum design in the RBDO procedure. The cost value and the probabilities of failure at the optimal designs are shown in Table 12. In the DDO process, the cost function is greatly reduced compared to the initial design (baseline design), while the probabilities of failure are not reduced as much. The probabilities of failure are reduced below the 10% target probability of failure after RBDO; at the same time, the cost function has a small increment. Again, the successful convergence verifies that the RBDO is performed effectively using the developed sensitivity method. Therefore, it can be concluded that the developed design sensitivity method performs well for RBDO with fixed COV.

Table 12 Cost and Probability of Failure at Optimal Designs

Design variable	Initial design	DDO	RBDO
Cost	269.47	257.58	259.88
G_1	48.8%	49.4%	9.9%
G_2	50.8%	50.9%	10.0%
G_3	53.9%	25.7%	0.0%
G_4	54.6%	16.9%	0.1%
G_5	58.4%	58.5%	1.8%
G_6	58.8%	65.4%	9.9%
G_7	58.9%	67.2%	9.9%
G_8	53.3%	55.9%	9.9%
G_9	51.5%	52.3%	6.9%
G_{10}	49.3%	0.1%	0.0%
G_{11}	51.0%	50.6%	10.0%

5. CONCLUSION

In many industrial problems, the tolerance of an input random variable is specified as a percentage value of the mean of the input random variable. To obtain reliable yet less costly design in the problems, RBDO with fixed COV should be considered. When the COV of an input random variable is fixed, the STDEV changes along with the design variable, which is the mean of the input random variable. Then, input randomness fluctuates due to the change of the STDEV. As a consequence, the optimum design of RBDO becomes a moving target problem due to the fluctuation when COV is fixed. Therefore, accurate design sensitivity of probability of failure is necessary to achieve a correct optimum design in RBDO with fixed COV.

In this paper, the design sensitivity of the probability of failure has been developed using a first-order score function for sampling-based RBDO with fixed COV. The score function contains the effects of changes in both mean and STDEV. Therefore, the developed design sensitivity can consider the change of the STDEV due to the change of design variable, which is the mean of the input random variable. Correlated input random variables are considered in the developed design sensitivity having a copula density function in the score function derivation.

Using a highly nonlinear function, the developed design sensitivity of the probability of failure has been tested using the FDM. The accuracy of the developed design sensitivity is verified as the sensitivity agrees with the FDM design sensitivity. Moreover, the developed method uses only 0.25% MCS samples of the FDM design sensitivity, and thus quite efficient. The RBDO with fixed COV is performed for a 2-D mathematical example using the developed design sensitivity. The optimization result shows smooth convergence, so the effectiveness of the developed method has been confirmed. In addition, for the same example, it is also shown that ignoring the effect of the STDEV in the design sensitivity causes difficulty in the convergence of the optimization. Hence, it can be seen that the developed design sensitivity is necessary for RBDO with fixed COV. Finally, the developed method is applied to a 14-D engineering example. The example converges in 19 design iterations with 11 additional line searches from DDO optimum design. This indicates that the developed sensitivity provided very accurate information in the optimization direction search and that the accurate design sensitivity helped the optimization process use few design iterations so the whole optimization process could be efficient. Hence, it is confirmed that the developed design sensitivity method is effective and efficient.

ACKNOWLEDGMENTS

Research is supported by the University of Iowa 2013 Ice Commercialization GAP Fund. This research was also partially supported by the Automotive Research Center (ARC) in accordance with Cooperative Agreement W56HZV-04-2-0001 U.S. Army Tank Automotive Research, Development and Engineering Center (TARDEC); and the Technology Innovation Program (10048305, Launching Plug-in Digital Analysis Framework for Modular System Design) funded by the Ministry of Trade, Industry & Energy (MI, Korea). These supports are greatly appreciated.

DISTRIBUTION STATEMENT A. Approved for public release; distribution is unlimited.

APPENDIX

Table A.1 Probability Density Function (PDF) and Cumulative Distribution Function (CDF) of Marginal Distribution

Distribution type	PDF $f_X(x;a,b)$	CDF $F_X(x;a,b)$	Ω^x
Normal	$\frac{1}{\sqrt{2\pi}b} e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2}$	$\Phi\left(\frac{x-a}{b}\right)$ ‡	$(-\infty,\infty)$
Lognormal	$\frac{1}{\sqrt{2\pi}xb} e^{-\frac{1}{2}\left(\frac{\ln x-a}{b}\right)^2}$	$\Phi\left(\frac{\ln x-a}{b}\right)$	$(0,\infty)$
Weibull	$\frac{b}{a}\left(\frac{x}{a}\right)^{b-1} e^{-\left(\frac{x}{a}\right)^b}$	$1-e^{-\left(\frac{x}{a}\right)^b}$	$(0,\infty)$
Gumbel	$\frac{1}{b}\exp\left[-\frac{x-a}{b}-\exp\left(-\frac{x-a}{b}\right)\right]$	$\exp\left[-\exp\left(-\frac{x-a}{b}\right)\right]$	$(-\infty,\infty)$
Gamma	$x^{a-1}\frac{e^{-x/b}}{\Gamma(a)b^a}$ §	$P(b,x/a)$ **	$(0,\infty)$
Extreme	$\frac{1}{b}\exp\left[\frac{x-a}{b}-\exp\left(\frac{x-a}{b}\right)\right]$	$1-\exp\left[-\exp\left(\frac{x-a}{b}\right)\right]$	$(-\infty,\infty)$
Extreme Type II	$\frac{a}{b}\left(\frac{b}{x}\right)^{a+1} e^{-\left(\frac{b}{x}\right)^a}$	$e^{-\left(\frac{b}{x}\right)^a}$	$(0,\infty)$

Table A. 2 Relationship between Mean, STDEV, and Parameters of Marginal Distribution

Distribution type	Parameters	Domain of parameters
Normal	$\mu=a,\ \sigma=b$	$a\in(-\infty,\infty),\ b\in(0,\infty)$
Lognormal	$\mu=e^{a+b^2/2}$ $\sigma^2=\left(e^{b^2}-1\right)e^{2a+b^2}$	$a\in(-\infty,\infty),\ b\in(0,\infty)$
Weibull	$\mu=a\Gamma(1+1/b)$ $\sigma^2=a^2[\Gamma(1+2/b)-\Gamma^2(1+1/b)]$	$a\in(0,\infty),\ b\in(0,\infty)$
Gumbel	$\mu=a+0.5772b,\ \sigma^2=b^2\pi^2/6$	$a\in(-\infty,\infty),\ b\in(0,\infty)$
Gamma	$\mu=ab,\ \sigma^2=ab^2$	$a\in(0,\infty),\ b\in(0,\infty)$
Extreme	$\mu=a-0.5772b,\ \sigma^2=b^2\pi^2/6$	$a\in(-\infty,\infty),\ b\in(0,\infty)$
Extreme Type II	$\mu=b\,\Gamma(1-1/a)$ $\sigma^2=b^2[\Gamma(1-2/a)-\Gamma^2(1-1/a)]$	$a\in(0,\infty),\ b\in(0,\infty)$

‡ $\Phi(\bullet)$: CDF of standard normal distribution

§ $\Gamma(\bullet)$: Gamma function, $\Gamma(s)\equiv\int_0^\infty t^{s-1}e^{-t}\,dt$

** $P(\bullet,\bullet)$: Reguralized gamma function, $P(s,x)\equiv\gamma(s,x)/\Gamma(s)=\int_0^x t^{s-1}e^{-t}\,dt/\Gamma(s)$

DISTRIBUTION STATEMENT A. Approved for public release; distribution is unlimited.

REFERENCES

[1] Tu, J., Choi, K. K., and Park, Y. H., 1999, "A New Study on Reliability-Based Design Optimization," *Journal of Mechanical Design*, **121**(4), pp. 557-564.

[2] Tu, J., Choi, K. K., and Park, Y. H., 2001, "Design Potential Method for Robust System Parameter Design," *AIAA Journal*, **39**(4), pp. 667-677.

[3] Lee, I., Choi, K. K., and Gorsich, D., 2010, "System Reliability-Based Design Optimization Using the MPP-Based Dimension Reduction Method," *Structural and Multidisciplinary Optimization*, **41**(6), pp. 823-839.

[4] Lee, I., Choi, K. K., Noh, Y., Zhao, L., and Gorsich, D., 2011, "Sampling-Based Stochastic Sensitivity Analysis Using Score Functions for RBDO Problems with Correlated Random Variables," *Journal of Mechanical Design*, **133**(2), pp. 021003-021003-10.

[5] Lee, I., Choi, K. K., and Zhao, L., 2011, "Sampling-Based RBDO Using the Stochastic Sensitivity Analysis and Dynamic Kriging Method," *Structural and Multidisciplinary Optimization*, **44**(3), pp. 299-317.

[6] Du, X., and Chen, W., 2004, "Sequential Optimization and Reliability Assessment Method for Efficient Probabilistic Design," *Journal of Mechanical Design*, **126**(2), pp. 225-233.

[7] Zou, T., and Mahadevan, S., 2006, "A Direct Decoupling Approach for Efficient Reliability-Based Design Optimization," *Structural and Multidisciplinary Optimization*, **31**(3), pp. 190-200.

[8] Kharmanda, G., Mohamed, A., and Lemaire, M., 2002, "Efficient Reliability-Based Design Optimization Using a Hybrid Space with Application to Finite Element Analysis," *Structural and Multidisciplinary Optimization*, **24**(3), pp. 233-245.

[9] Liang, J., Mourelatos, Z. P., and Nikolaidis, E., 2007, "A Single-Loop Approach for System Reliability-Based Design Optimization," *Journal of Mechanical Design*, **129**(12), pp. 1215-1224.

[10] Wang, B. P., 2004, "Mean Value Method for Reliability Based Optimization Revisited," *SAE Technical Paper*, 2004-01-1126.

[11] Hasofer, A. M., and Lind, N. C., 1974, "Exact and Invariant Second-Moment Code Format," *Journal of the Engineering Mechanics Division, ASCE*, **100**(1), pp. 111-121.

[12] Haldar, A., and Mahadevan, S., 2000, *Probability, Reliability and Statistical Methods in Engineering Design*, John Wiley & Sons, New York, NY.

[13] Hohenbichler, M., and Rackwitz, R., 1988, "Improvement of Second-Order Reliability Estimates by Importance Sampling," *Journal of Engineering Mechanics*, **114**(12), pp. 2195-2199.

[14] Breitung, K., 1984, "Asymptotic Approximations for Multinormal Integrals," *Journal of Engineering Mechanics*, **110**(3), pp. 357-366.

[15] Rahman, S., and Wei, D., 2006, "A Univariate Approximation at Most Probable Point for Higher-Order Reliability Analysis," *International Journal of Solids and Structures*, **43**(9), pp. 2820-2839.

[16] Hou, G. J. W., 2004, "A Most Probable Point-Based Method for Reliability Analysis, Sensitivity Analysis, and Design Optimization," *NASA*, NASA/CR-2004-213002.

[17] Yin, X., and Chen, W., 2006, "Enhanced Sequential Optimization and Reliability Assessment Method for Probabilistic Optimization with Varying Design Variance," *Structure and Infrastructure Engineering*, **2**(3-4), pp. 261-275.

[18] Noh, Y., Choi, K. K., and Du, L., 2009, "Reliability-Based Design Optimization of Problems with Correlated Input Variables Using a Gaussian Copula," *Structural and Multidisciplinary Optimization*, **38**(1), pp. 1-16.

[19] Noh, Y., Choi, K. K., and Lee, I., 2010, "Identification of Marginal and Joint CDFs Using Bayesian Method for RBDO," *Structural and Multidisciplinary Optimization*, **40**(1-6), pp. 35-51.

[20] Rubinstein, R. Y., and Kroese, D. P., 2008, *Simulation and the Monte Carlo Method, 2nd Ed.*, John Wiley & Sons, Hoboken, NJ.

[21] Rubinstein, R. Y., and Shapiro, A., 1993, *Discrete Event Systems: Sensitivity Analysis and Stochastic Optimization by the Score Function Method*, Wiley, New York, NY.

[22] Rahman, S., 2009, "Stochastic Sensitivity Analysis by Dimensional Decomposition and Score Functions," *Probabilistic Engineering Mechanics*, **24**(3), pp. 278-287.

[23] Nelsen, R. B., 2006, *An Introduction to Copulas*, *2nd Ed.*, Springer, New York, NY.

[24] Geddes, K. O., Glasser, M. L., Moore, R. A., and Scott, T. C., 1990, "Evaluation of Classes of Definite Integrals Involving Elementary Functions via Differentiation of Special Functions," *Applicable Algebra in Engineering, Communication and Computing*, **1**(2), pp. 149-165.

[25] Cho, H., Bae, S., Choi, K. K., Lamb, D., and Yang, R.-J., 2014, "An Efficient Variable Screening Method for Effective Surrogate Models for Reliability-Based Design Optimization," *Structural and Multidisciplinary Optimization*, **50**(5), pp. 717-738.

DISTRIBUTION STATEMENT A. Approved for public release; distribution is unlimited.